MODULI OF CURVES

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Abstract. We treat some general facts about moduli spaces, in particular moduli of curves of genus g. We give some examples, and construct coarse moduli spaces for curves of genus zero and elliptic curves. We introduce briefly the concept of algebraic stack. The main goal is the proof of the smoothness of the Deligne-Mumford stack which parametrizes curves of genus $g \geq 2$, in the proof we use some results from deformation theory.

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Introduction

The search for an object M_g that parametrizes curves of genus g is a very classical problem in Algebraic Geometry. Riemann was the first to calculate the number $3g-3$, which gives the dimension of M_g. Most of the moduli space of curves are only coarse moduli spaces, and the obstructions to representing the corresponding functors, i.e to find fine moduli spaces, come from automorphisms of the objects we want to parametrize.
The fact that a moduli functor does not admit a fine moduli space means that it cannot be represented in the category of schemes. A larger category is that of functors from schemes to sets, by definition moduli functors are objects in it, but in this way we lose completely the geometric intuition. Categories of Algebraic Stacks are other enlargements of the category of schemes that have been used to study moduli problems.
The moduli space M_g is a coarse moduli space for smooth, complete, connected curve of genus g over an algebraically closed field. This space can be constructed using different techniques, as instance: the Teichmüller approach, the Hodge theory approach and the Geometric Invariant Theory (G.I.T.) approach.
The book *Geometric Invariant Theory* by Mumfor contains the proof of existence of a coarse moduli

Date: April-May 2010.

space for curves of genus g. Deligne and Mumford proved the irreducibility of the compactification of the moduli space, and introduced the language of stacks in the article *The irreducibility of the space of curves of given genus* published in 1969.
In this work we give an introduction to moduli questions, we introduce some tools, like spectral sequences and deformation theory. We discuss briefly moduli questions for curves of genus zero and elliptic curves. Finally we define the moduli functor for curves of genus $g \geq 2$, assuming that this functor is represented by an algebraic stack $\overline{\mathfrak{M}_g}$, and using deformation theory we will prove that $\overline{\mathfrak{M}_g}$ is a smooth Deligne-Mumford stack.

1. Moduli Questions

To fix the ideas, we work over an algebraically closed field k. Consider a class of objects $\mathcal{M}$ aver k, as instance the class of closed subschemes of $\mathbb{P}^n$ with fixed Hilbert Polynomial, the class of curves of genus g over k, the class of vector bundles of given rank and Chern classes over a fixed schemes, and so on. We wish to classify the objects in $\mathcal{M}$.
The first step is to give a rule for saying when two objects of $\mathcal{M}$ are the same (usually isomorphic) and then to give the elements of $\mathcal{M}$ up to isomorphism. This determines $\mathcal{M}$ as a set. Now we want to put a natural structure of variety or scheme on $\mathcal{M}$. In other words we are looking for a scheme M whose closed points are in one-to-one correspondence with the elements of $\mathcal{M}$, and whose scheme structure describes the variations of elements in $\mathcal{M}$, more precisely how they behave in families.

Definition 1.1. A family of elements of $\mathcal{M}$, over the parameter scheme S of finite type over k, is a scheme $X \to S$ flat over S, whose fibers at closed points are elements of $\mathcal{M}$.

The first request on M, to be a Moduli Space for the class $\mathcal{M}$, is that for any family $X \to S$ of objects of $\mathcal{M}$ there exists a morphism $\phi : S \to M$ such that for any closed point $s \in S$, the image $f(s) \in \mathcal{M}$ corresponds to the isomorphism class of the fiber $X_s = \phi^{-1}(s)$ in $\mathcal{M}$.
Furthermore we want the assignment of the morphism ϕ to be functorial. To explain the last sentence consider the functor $\mathcal{F} : \mathfrak{Sch} \to \mathfrak{Sets}$, that assigns to S the set $\mathcal{F}(S)$ of families $X \to S$ of elements of $\mathcal{M}$ parametrized by S. If $S^{'} \to S$ is a morphism, for any family $X \to S$ we can consider the fiber product $X \times_S S^{'} \to S^{'}$, that is a family over $S^{'}$. In this way the morphism $S^{'} \to S$ gives rise to a map of set $\mathcal{F}(S) \to \mathcal{F}(S^{'})$, and $\mathcal{F}$ becomes a controvariant functor.
In this language to assign a morphism $\phi : S \to M$ to any family $X \to S$ with the required properties, means to give a functorial morphism $\alpha : \mathcal{F} \to Hom(-, M)$.
Finally we want to make M unique with the above properties. So we require that if N is any other scheme, and $\beta : \mathcal{F} \to Hom(-, N)$ is a functorial morphism, then there exists a unique morphism $e : M \to N$ such that $\beta = h_e \circ \alpha$, where $h_e : Hom(-, M) \to Hom(-, N)$ is the induced map on associated functors.

Definition 1.2. We define a *coarse moduli space* for the family $\mathcal{M}$ to be a scheme M over k, with a morphism of functors $\alpha : \mathcal{F} \to Hom(-, M)$ such that

- the induced map $\mathcal{F}(Spec(k)) \to Hom(Spec(k), M)$ is bijective i.e. there is a one-to-one correspondence with isomorphism classes of elements of $\mathcal{M}$ and closed points of M,
- α is universal in the sense explained above.

We define a *tautological family* for $\mathcal{M}$ to be a family $X \to M$ such that for each closed point $m \in M$, the fiber X_m is the element of $\mathcal{M}$ corresponding to m by the bijection $\mathcal{F}(Spec(k)) \to Hom(Spec(k), M)$ above.

A *jump phenomenon* for $\mathcal{M}$ is a family $X \to S$, where S is an integral scheme of dimension at least one, such that all fibers X_s for $s \in S$ are isomorphic except for one X_{s_0} that is different. In this case the corresponding morphism $S \to M$ have to map s_0 to a point and all other closed points of S to another point, but this is not possible for a morphism of schemes, so a coarse moduli space for $\mathcal{M}$ fails to exist.

Example 1.3. *Consider the family $y^2 = x^3 + t^2x + t^3$ over the t-line. Then for any $t \neq 0$ we get smooth elliptic curves all with the same j-invariant*

$$j = 12^3 \cdot \frac{4t^6}{4t^6 + 27t^6} = 12^3 \cdot \frac{4}{31},$$

and hence all isomorphic. But for $t = 0$ we get the cusp $y^2 = x^3$. This is a jump phenomenon, so the cuspid curve cannot belong to a class having a coarse moduli space.

Definition 1.4. Let $\mathcal{F}$ be the functor associated to the moduli problem $\mathcal{M}$. If $\mathcal{F}$ is isomorphic to a functor of the form $Hom(-, M)$, the we say that $\mathcal{F}$ is representable, and we call M a *fine moduli space* for $\mathcal{M}$.

Let $\alpha : \mathcal{F} \to Hom(-, M)$ be an isomorphism. In particular $\mathcal{F}(M) \to Hom(M, M)$ is an isomorphism, and there is a unique family $X_{\mathcal{U}} \to M$ corresponding to the identity map $Id_M \in Hom(M, M)$. The family $X_{\mathcal{U}}$ is called the *universal family* of the fine moduli space M. Note that for any family $X \to S$ there exists an unique morphism $S \to M$, such that $X \to S$ is obtained by base extension from the universal family. Conversely, if there is a scheme M and a family $X_{\mathcal{U}}$ with the above properties then $\mathcal{F}$ is represented by M.

Remark 1.5. If M is a fine moduli space for $\mathcal{M}$ then it is also a coarse moduli space, furthermore the universal family $X_{\mathcal{U}} \to M$ is a tautological family.

A benefits of having a fine moduli space is that we can study it using infinitesimal methods.

Proposition 1.6. *Let M be a fine moduli space for the moduli problem $\mathcal{M}$, and let $X_0 \in \mathcal{M}$ be an element corresponding to a point $x_0 \in M$. The Zariski tangent space $T_{x_0}M$ is in one-to-one correspondence with the set of families $X \to D$ over the dual numbers $D = k[\epsilon]/(\epsilon^2)$, whose closed fibers are isomorphic to X_0.*

Proof. We know that to give a morphism $f : Spec(D) \to M$ is equivalent to give a closed point $x_0 \in M$ and a tangent direction $v \in T_{x_0}M$. But a morphism $f : Spec(D) \to M$ corresponds to a unique family $X \to Spec(D)$ whose closed fibers are isomorphic to $X_0 \in \mathcal{M}$ corresponding to the point $x_0 \in M$, where $x_0 = f((Spec(D))_{red})$. $\square$

Let $\mathcal{F} : \mathfrak{Sch} \to \mathfrak{Sets}$ be the functor associated to the moduli problem $\mathcal{M}$. Suppose that $\mathcal{F}$ is representable, and let M be the corresponding fine moduli space. For any local Artin k-algebra A we have that $Spec(A)$ is a fat point and $(Spec(A))_{red}$ is a single point. For any $x_0 \in M$ we can define the infinitesimal deformation functor of $\mathcal{F}$ as the functor $\mathfrak{Art} \to \mathfrak{Sets}$ that sends A in the set of morphism $f : Spec(A) \to M$ such that $f((Spec(A))_{red}) = x_0$. Clearly studying this functor we get information on the geometry of M in a neighborhood of x_0.
Recall that a pro-object is an inverse limit of objects in $\mathfrak{Art}$, the category of Artin local algebras over a field k. If $\mathcal{F} : \mathfrak{Art} \to \mathfrak{Sets}$ is a deformation functor we say that $\mathcal{F}$ is pro-representable if it is isomorphic to $Hom(-, R)$ for some pro-object R.

Proposition 1.7. *Let $\mathcal{F}$ be the functor associated to the moduli problem $\mathcal{M}$, and $X_0 \in \mathcal{M}$. Consider the functor $\mathcal{F}_0$ that to each local Artin ring A over k assigns the set of families of $\mathcal{M}$ over $Spec(A)$ whose closed fiber is isomorphic to X_0. If $\mathcal{M}$ has a fine moduli space, then the functor $\mathcal{F}_0$ is pro-representable.*

Proof. Let M be a fine moduli scheme for $\mathcal{M}$, and let $x_0 \in M$ corresponds to $X_0 \in \mathcal{M}$. Let $\mathcal{O}_{M,x_0}$ be the local ring of M at x_0 and $\mathfrak{M}_{x_0}$ its maximal ideal. The natural homomorphisms

$$... \to \mathcal{O}_{M,x_0}/\mathfrak{M}_{x_0}^3 \to \mathcal{O}_{M,x_0}/\mathfrak{M}_{x_0}^2 \to \mathcal{O}_{M,x_0}/\mathfrak{M}_{x_0},$$

make $(\mathcal{O}_{M,x_0}/\mathfrak{M}^n_{x_0})$ into an inverse system of rings. The inverse limit $\varprojlim \mathcal{O}_{M,x_0}/\mathfrak{M}^n_{x_0}$ is denoted by $\hat{\mathcal{O}}_{M,x_0}$, and is called the completion of $\mathcal{O}_{M,x_0}$ with respect to $\mathfrak{M}_{x_0}$ or the $\mathfrak{M}_{x_0}$-adic completion of $\mathcal{O}_{M,x_0}$.
Since M is a fine moduli space, each element of $\mathcal{F}_0(A)$ corresponds to a unique morphism $Spec(A) \to M$ that maps $(Spec(A)_{red}) = Spec(k)$ at x_0. Such morphism correspond to a ring homomorphism $\hat{\mathcal{O}}_{M,x_0} \to A$. We conclude that the functor $\mathcal{F}_0$ is pro-representable and that it is represented by the pro-object $\hat{\mathcal{O}}_{M,x_0}$, $\mathfrak{M}_{x_0}$-adic completion of $\mathcal{O}_{M,x_0}$. □

Definition 1.8. A controvariant functor $\mathcal{F} : \mathfrak{Sch} \to \mathfrak{Sets}$ is a *sheaf for the Zariski topology*, if for every scheme S and every $\{\mathcal{U}_i\}$ open covering of S, the diagram

$$\mathcal{F}(S) \to \prod \mathcal{F}(\mathcal{U}_i) \rightrightarrows \prod \mathcal{F}(\mathcal{U}_i \cap \mathcal{U}_j)$$

is exact. This means that:

- given $x, y \in \mathcal{F}(S)$ whose restriction to $\mathcal{F}(\mathcal{U}_i)$ are equal for all i, then $x = y$,
- given a collection of elements $x_i \in \mathcal{F}(\mathcal{U}_i)$ for each i, such that for each i, j, the restrictions of x_i, x_j to $\mathcal{F}(\mathcal{U}_i \cap \mathcal{U}_j)$ are equal, then there exists an element $x \in \mathcal{F}(S)$ whose restriction to each $\mathcal{F}(\mathcal{U}_i)$ is x_i.

Proposition 1.9. *If the moduli problem $\mathcal{M}$ has a fine moduli space, then the associated functor $\mathcal{F}$ is a sheaf in the Zariski topology.*

Proof. Since $\mathcal{M}$ has a fine moduli space, for any scheme S we have $\mathcal{F}(S) = Hom(S, M)$. Furthermore a morphisms of schemes are determined locally, and can be glued if they are given locally and are compatible on overlaps. □

Remark 1.10. Using Grothendieck's theory of *descent* one can show that a representable functor is a sheaf for the *faithfully flat quasi-compact topology*, and hence also for the *étale topology*.

1.1. **Examples of Moduli Spaces.** We will give some examples of representable functors.

Example 1.11. *(<u>Grassmannians</u>) Let V be a k-vector space of dimension n, and let $r \leq n$ be a fixed integer. Consider the controvariant functor $Gr : \mathfrak{Sch} \to \mathfrak{Sets}$ defined as follows*

- *For any scheme S, $Gr(S)$ is the set of rank r vector subbundle of the trivial bundle $S \times V$.*
- *If $f : S \to S'$ is a morphism of schemes, and $E_{S'}$ is a rank r subbundle of $S' \times V$, we define*

$$Gr(f)(E_{S'}) = f^*(E_{S'}) = (f \times Id_V)^{-1}(E_{S'}).$$

Note that for $S = Spec(k)$ we have that $Gr(Spec(k))$ is the set of rank r subbundle of $Spec(k) \times V = V$ i.e. the set of r-dimensional subspace of V, that is the Grassmannian $Gr(r, V)$.
If $E \in Gr(S)$ is a rank r subbundle of $S \times V$, we can construct a morphism $f_E : S \to Gr(r, V)$ defined by $s \mapsto E_s$, where E_s is the fiber of E over $s \in S$. In this way we get a map

$$\phi(S) : Gr(S) \to Hom(S, Gr(r, V)),\ E \mapsto f_E.$$

The collection $\{\phi(S)\}$ gives a functorial isomorphism between Gr and $Hom(-, Gr(r, V))$. Then the functor Gr is representable and the Grassmannian $Gr(r, V)$ is the corresponding fine moduli space. The universal family corresponding to the identity map $Id_{Gr(r,V)} \in Hom(Gr(r, V), Gr(r, V))$ is clearly the universal bundle on $Gr(r, V)$ given by $\{(W, v) \mid v \in W\} \subseteq Gr(r, V) \times V$.

Example 1.12. *(<u>Hilbert Scheme</u>) Let $P \in \mathbb{Q}[z]$ be a fixed polynomial. For any S scheme over k consider $\mathbb{P}^N_S = \mathbb{P}^N \times_k S$, and the functor*

$$Hilb^N_P : \mathfrak{Sch} \to \mathfrak{Sets},$$

that maps S in the set of subschemes $Y \subseteq \mathbb{P}^N_S$ such that the projection $\pi : Y \to S$ is flat, and for any $s \in S$ the fiber $\pi^{-1}(s)$ is a subscheme of $\mathbb{P}^N$ with Hilbert polynomial P. The functor $Hilb^N_P$ is representable by a scheme $Hilb_P(\mathbb{P}^N)$ projective over k and called the Hilbert Scheme.

To any closed schemes $Y \subseteq \mathbb{P}^N$ we can associate its structure sheaf $\mathcal{O}_Y$, its ideal sheaf $\mathcal{I}_Y$, and the structure sequence

$$0 \mapsto \mathcal{I}_Y \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_Y \mapsto 0.$$

Then we can regard the Hilbert scheme as the space parametrizing all the quotients $\mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_Y$, with Hilbert polynomial P.

Example 1.13. *(Grothendieck's Quot Scheme) As a generalization of the discussion above consider a fixed coherent sheaf $\mathcal{E}$ on $\mathbb{P}^N$. The scheme parametrizing all the quotients $\mathcal{E} \to \mathcal{F} \mapsto 0$ with Hilbert polynomial P is called the Quot Scheme. Grothendieck showed that the local deformation functor of the Quot functor is pro-representable and that the Quot functor is representable by a projective scheme.*

Example 1.14. *(Picard Scheme) Let X be a scheme of finite type over an algebraically closed field k and let $x \in X$ be a fixed point. Consider the functor*

$$Pic_{X,x} : \mathfrak{Sch} \to \mathfrak{Sets},$$

that associates to S the group of all invertible shaves $\mathcal{L}$ on $X \times S$, with a fixed isomorphism $\mathcal{L}_{|x} \times S \cong \mathcal{O}_S$.
If X is integral and projective, then this functor is representable by a separated scheme, locally of finite type over k, called the Picard Scheme of X.

Example 1.15. *(Hilbert-Flag Scheme) Consider a functor that associates to each scheme S a flag $Y_1 \subseteq Y_2 \subseteq ... \subseteq Y_k \subseteq \mathbb{P}^N_S$ of closed subscheme, all flat over S and where the fibers if Y_j have a fixed Hilbert Polynomial P_j for any $j = 1, ..., k$. This functor is representable by a scheme, projective over k, called the Hilbert-Flag Scheme.*

2. Curves of Genus Zero

There is only one smooth curve of genus $g = 0$ over an algebraically closed field k, namely $\mathbb{P}^1_k$. A family of curve of genus zero over a scheme S is a scheme X, smooth and projective over S, whose fibers are curves of genus zero.

Proposition 2.1. *The space $M = Spec(k)$ is a coarse moduli scheme for curves of genus zero. Furthermore it has a tautological family.*

Proof. The set $Hom(Spec(k), Spec(k))$ consists of a single element and clearly is in one-to-one correspondence with the set of families over $Spec(k)$ that consists of the family $\mathbb{P}^1_k \to Spec(k)$. Clearly $\mathbb{P}^1_k \to Spec(k)$ is a tautological family. If $X \to S$ is a family there is a unique morphism $S \to M = Spec(k)$, in this way we get the functorial morphism $\alpha : \mathcal{F} \to Hom(-, M)$.
Now suppose that $\beta : \mathcal{F} \to Hom(-, N)$ is another morphism of functors. In particular the family $\mathbb{P}^1_k \to M$ determines a morphism $e \in Hom(M, N)$. Let $X \to S$ a family over a scheme S of finite type over k. For any closed point $s \in S$ the fiber is $X_s \cong \mathbb{P}^1$, then any closed point s goes to the point $n = e(M) \in N$. Now the restriction of the family on S to an Artin closed subscheme of S is trivial, so factor through $Spec(k)$. We conclude that the morphism β factors through α. □

Clearly the tautological family is $\mathbb{P}^1 \to Spec(k)$, that is the unique family over $M = Spec(k)$. Suppose $M = Spec(k)$ to be a fine moduli space for the curves of genus zero. Then the universal family is $\mathbb{P}^1 \to Spec(k)$. Since any other family is obtained by base extension from the universal family it must be trivial i.e. of the form $\mathbb{P}^1 \times_k S \to S$. But the ruled surfaces provide an example of non trivial families of curves of genus zero.

Consider as instance the blow up $Bl_p\mathbb{P}^2$ of $\mathbb{P}^2$ is a point p. The projection $\pi : Bl_p\mathbb{P}^2 \to \mathbb{P}^1$ makes $Bl_p\mathbb{P}^2$ into a ruled surface, but it is not a product. Note that $Pic(Bl_p\mathbb{P}^2) = Pic(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z}\oplus\mathbb{Z}$, but on $Bl_p\mathbb{P}^2$ we have a (-1)-curve, the exceptional divisor. Suppose that there is a (-1)-curve $C = (a, b)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. We have $C^2 = (aL + bR)(aL + bR) = 2ab = -1$, a contradiction.

Definition 2.2. A *pointed* curve of genus zero over k is a curve of genus zero with a choice of a k-rational point. A family of pointed curves of genus zero is a flat family $X \stackrel{\pi}{\to} S$, whose geometric fibers are curves of genus zero, with a section $\sigma : S \to X$.

The fact that $\sigma : S \to X$ is a section means that $\pi \circ \sigma = Id_S$. Then for any point $s \in S$ the image $\sigma(s)$ is a point of the fiber $X_s \cong \mathbb{P}^1$ over s. The section σ is sometimes called an S-point of X.

A way to obtain a fine moduli space for the curves of genus zero is to rigidify the curves by taking three distinct points. We know that there is a unique automorphism of $\mathbb{P}^1$ that fixed three distinct points, namely the identity. Consider the families of curves of genus zero with three marked points i.e. the families of $X \to S$, whose fibers are curves of genus zero, with three sections $\sigma_1, \sigma_2, \sigma_3 : S \to X$, such that on each fiber the sections have distinct support. Since a curve X of genus zero with three marked points is rigid i.e. $Aut(X) = \{Id_X\}$, the corresponding functor is representable by $M = Spec(k)$ and the universal family is $\mathbb{P}^1 \to Spec(k)$ with three distinct points, say $[0 : 1], [1 : 0], [1 : 1]$.

3. Grothendieck Spectral Sequence

We begin recalling the notion of *five terms exact sequence* or *exact sequence of low degree terms* associated to a spectral sequence. Let

$$E_2^{h,k} \Longrightarrow H^n(A)$$

be a spectral sequence whose terms are non trivial only for $h, k \geq 0$. Then these is an exact sequence

$$0 \mapsto E_2^{1,0} \to H^1(A) \to E_2^{0,1} \to E_2^{2,0} \to H^2(A).$$

The *Grothendieck spectral sequence* is an algebraic tool to express the derived functors of a composition of functors $\mathcal{G} \circ \mathcal{F}$ in terms of the derived functors of $\mathcal{F}$ and $\mathcal{G}$.

Let $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$ and $\mathcal{G} : \mathcal{C}_2 \to \mathcal{C}_3$ be two additive covariant functors between abelian categories. Suppose that $\mathcal{G}$ is left exact and that $\mathcal{F}$ takes injective objects of $\mathcal{C}_1$ in $\mathcal{G}$-acyclic objects of $\mathcal{C}_2$. Then there exists a spectral sequence for any objects A of $\mathcal{C}_1$

$$E_2^{h,k} = (R^h\mathcal{G} \circ R^k\mathcal{F})(A) \Longrightarrow R^{h+k}(\mathcal{G} \circ \mathcal{F})(A).$$

The corresponding exact sequence of low degrees is the following

$$0 \mapsto R^1\mathcal{G}(\mathcal{F}(A)) \to R^1(\mathcal{G}\mathcal{F}(A)) \to \mathcal{G}(R^1\mathcal{F}(A)) \to R^2\mathcal{G}(\mathcal{F}(A)) \to R^2(\mathcal{G}\mathcal{F})(A).$$

As a special case of the Grothendieck spectral sequence we get the *Leray spectral sequence*. Let $f : X \to Y$ be a continuous map between topological spaces. We take $\mathcal{C}_1 = \mathfrak{Ab}(X)$ and $\mathcal{C}_2 = \mathfrak{Ab}(Y)$ be the categories of shaves of abelian groups over X and Y respectively. The we take $\mathcal{F}$ to be the direct image functor $f_* : \mathfrak{Ab}(X) \to \mathfrak{Ab}(Y)$ and $\mathcal{G} = \Gamma_Y : \mathfrak{Ab}(Y) \to \mathfrak{Ab}$ be the global section functor, where $\mathfrak{Ab}$ is the category of abelian groups. Note that

$$\Gamma_Y \circ f_* = \Gamma_X : \mathfrak{Ab}(X) \to \mathfrak{Ab}$$

is the global section functor on X. By Grothendieck spectral sequence we know that $(R^h\Gamma_Y \circ R^kf_*)(\mathcal{E}) \Longrightarrow R^{h+k}(\Gamma_Y \circ f_*)(\mathcal{E}) = R^{h+k}\Gamma_X(\mathcal{E})$ for any $\mathcal{E} \in \mathfrak{Ab}(X)$, that is

$$H^h(Y, R^kf_*\mathcal{E}) \Longrightarrow H^{h+k}(X, \mathcal{E}).$$

The exact sequence of low degrees looks like

$$0 \mapsto H^1(Y, f_*\mathcal{E}) \to H^1(X, \mathcal{E}) \to H^0(Y, R^1 f_*\mathcal{E}) \to H^2(Y, f_*\mathcal{E}) \to H^2(X, \mathcal{E}).$$

Finally we work out the *spectral sequence of Ext functors.* Let $\mathcal{E} \in \mathfrak{Coh}(X)$ be a coherent sheaf on a scheme X. Consider the functor

$$\mathcal{H}om(\mathcal{E}, -) : \mathfrak{Coh}(X) \to \mathfrak{Coh}(X),\ \mathcal{Q} \mapsto \mathcal{H}om(\mathcal{E}, \mathcal{Q}),$$

and the global section functor

$$\Gamma_X : \mathfrak{Coh}(X) \to \mathfrak{Ab},\ \mathcal{Q} \mapsto \Gamma_X(\mathcal{Q}).$$

Note that $\Gamma_X \circ \mathcal{H}om(\mathcal{E}, -) = Hom(\mathcal{E}, -)$. By Grothendieck spectral sequence we have $(R^h\Gamma_X \circ R^k\mathcal{H}om(\mathcal{E}, -))(\mathcal{Q}) \Longrightarrow R^{h+k}(Hom(\mathcal{E}, -)(\mathcal{Q})$ for any $\mathcal{Q} \in \mathfrak{Coh}(X)$, that is

$$H^h(X, \mathcal{E}xt^k(\mathcal{E}, \mathcal{Q})) \Longrightarrow Ext^{h+k}(\mathcal{E}, \mathcal{Q}).$$

The corresponding sequence of low degrees is

$$0 \mapsto H^1(X, \mathcal{H}om(\mathcal{E}, \mathcal{Q})) \to Ext^1(\mathcal{E}, \mathcal{Q}) \to H^0(X, \mathcal{E}xt^1(\mathcal{E}, \mathcal{Q})) \to H^2(X, \mathcal{H}om(\mathcal{E}, \mathcal{Q})) \to Ext^2(\mathcal{E}, \mathcal{Q}).$$

4. Deformations of Schemes

Let X be a smooth scheme of finite type over k. We define the deformation functor $Def_X : \mathfrak{Art} \to \mathfrak{Sets}$ of X sending an Artin ring A to the set of couples $(X_A \stackrel{\pi_A}{\to} Spec(A), \phi)$ modulo isomorphism, where π_A is a smooth morphism, $\phi : X \to X_0$ is an isomorphism, X_0 is defined by the cartesian diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_A \arrow[d] \\ Spec(k) \arrow[r] & Spec(A) \end{tikzcd}$$

and (X_A, ϕ), $(X_A^{'}, \phi^{'})$ are isomorphic if there is an isomorphism $\alpha : X_A \to X_A^{'}$ such that the diagram

$$\begin{tikzcd} X_A \arrow[rr, "\alpha"] \arrow[dr, "\pi_A"'] & & X_A^{'} \arrow[dl, "\pi_A^{'}"] \\ & Spec(A) & \end{tikzcd}$$

commutes and $\phi^{'} = \alpha \circ \phi$.

Theorem 4.1. *For any semi-small exact sequence $0 \mapsto I \to A \to B \mapsto 0$ in $\mathfrak{Art}$, let $T^iDef_X = H^i(X, T_X)$, then*

(1) *there exists a functorial exact sequence*

$$T^1Def_X \otimes I \to Def_X(A) \to Def_X(B) \to T^2Def_X \otimes I;$$

(2) *for any $(X_A, \pi_A, \phi) \in Def_X(A)$, let $G = Stab(X_A) \subseteq T^1Def_X \otimes I$, we have a functorial exact sequence*

$$0 \mapsto T^0Def_X \otimes I \to Aut(X_A) \to Aut(X_B) \to G \mapsto 0.$$

Now let X be any scheme over k. Consider the exact sequence of low degree for Ext functors with shaves Ω_X and $\mathcal{O}_X$. We have

$$0 \mapsto H^1(X, \mathcal{H}om(\Omega_X, \mathcal{O}_X)) \to Ext^1(\Omega_X, \mathcal{O}_X) \to H^0(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) \to H^2(X, \mathcal{H}om(\Omega_X, \mathcal{O}_X)).$$

Identifying the set of deformations of X over the dual numbers $D = \frac{k[\epsilon]}{\epsilon^2}$ is in one-to-one correspondence with the group $Ext^1(\Omega_X, \mathcal{O}_X)$. Then we get the sequence

$$0 \mapsto H^1(X, \mathcal{H}om(\Omega_X, \mathcal{O}_X)) \to Def_X(D) \to H^0(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) \to H^2(X, \mathcal{H}om(\Omega_X, \mathcal{O}_X)).$$

5. Differentials and Ext groups

Let X be a smooth scheme and let Y be a closed subscheme with ideal sheaf $\mathcal{I}$. We have an exact sequence of shaves

$$\mathcal{I}/\mathcal{I}^2 \to \Omega_X \otimes \mathcal{O}_Y \to \Omega_Y \mapsto 0,$$

where the first map is the differential. Furthermore Y is smooth if and only if

- Ω_Y is locally free,
- the sequence is exact on the left also

$$0 \mapsto \mathcal{I}/\mathcal{I}^2 \to \Omega_X \otimes \mathcal{O}_Y \to \Omega_Y \mapsto 0.$$

In this case the sheaf $\mathcal{I}$ is locally generated by $Codim(Y, X)$ elements, and its is locally free of rank $Codim(Y, X)$ on Y.

Remark 5.1. Let $Y \subseteq X$ be an hypersurface not necessarily smooth. We can associate to Y a Cartier divisor $\{(\mathcal{U}_i, f_i)\}$, and the ideal sheaf $\mathcal{I}$ is locally generated by f_i on $\mathcal{U}_i$. Furthermore $\mathcal{O}_X(Y)$ is the sheaf locally generated by f_i^{-1} on $\mathcal{U}_i$. We conclude that $\mathcal{O}_X(-Y) \cong \mathcal{I}$ is locally free. If $Y \subseteq X$ is a reduced hypersurface, then $\mathcal{I}$ is locally free of rank one. We have the differential $d : \mathcal{I}/\mathcal{I}^2 \to \Omega_X \otimes \mathcal{O}_Y$, if f is a local generator of $\mathcal{I}$ then df is a local generator of $Im(d)$, since Y is reduced then $df \neq 0$, $Im(d)$ is locally free of rank one, and the map d is injective. So we have again an exact sequence

$$0 \mapsto \mathcal{I}/\mathcal{I}^2 \to \Omega_X \otimes \mathcal{O}_Y \to \Omega_Y \mapsto 0.$$

Let $f = f(x_1, ..., x_n)$, with $n = dim(X)$, be a local equation for Y in X. Then $df = \frac{\partial f}{\partial x_1} dx_1 + ... + \frac{\partial f}{\partial x_n}$. Since Y is reduced the differential is injective, furthermore $\mathcal{I}/\mathcal{I}^2$ is locally free of rank one and $\Omega_X \otimes \mathcal{O}_Y$ is locally free of rank n. Applying $Hom(-, \mathcal{O}_Y)$ to the sequence

$$0 \mapsto \mathcal{I}/\mathcal{I}^2 \to \Omega_X \otimes \mathcal{O}_Y \to \Omega_Y \mapsto 0,$$

we obtain

$$0 \mapsto Hom(\Omega_Y, \mathcal{O}_Y) \to Hom(\Omega_{X|Y}, \mathcal{O}_Y) \to Hom(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \to Ext^1(\Omega_Y, \mathcal{O}_Y) \to Ext^1(\Omega_{X|Y}, \mathcal{O}_Y).$$

Remark 5.2. Let X be a noetherian scheme such that any coherent shaves on X is quotient of a locally free shaves i.e. $Coh(X)$ has enough locally free objects. We define the homological dimension of $\mathcal{F} \in Coh(X)$, denoted by $hd(\mathcal{F})$, to be the least length of a locally free resolution of $\mathcal{F}$ or ∞ if there is no finite one. Clearly $\mathcal{F}$ is locally free if and only if $hd(\mathcal{F}) = 1$ if and only if $Ext^1(\mathcal{F}, \mathcal{G}) = 0$ far any $\mathcal{G} \in Mod(X)$. Furthermore $hd(\mathcal{F}) \leq n$ if and only if $Ext^i(\mathcal{F}, \mathcal{G}) = 0$ for any $i > n$ and $\mathcal{G} \in Mod(X)$. Finally $hd(\mathcal{F}) = Sup_{x \in X}(pd_{\mathcal{O}_x} \mathcal{F}_x)$, where pd is the projective dimension.

In our case $\Omega_{X|Y}$ is locally free, and by the preceding remark $Ext^1(\Omega_{X|Y}, \mathcal{O}_Y) = 0$. Then we get the exacts sequence

$$0 \mapsto Hom(\Omega_Y, \mathcal{O}_Y) \to Hom(\Omega_{X|Y}, \mathcal{O}_Y) \to Hom(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \to Ext^1(\Omega_Y, \mathcal{O}_Y) \mapsto 0.$$

Consider now the special case $X = \mathbb{A}^n$ and $Y = Spec(A)$, where $A = k[x_1, ..., x_n]/(f)$. The map $Hom(\Omega_{\mathbb{A}^n|Y}, \mathcal{O}_Y) \to Hom(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$ is the transpose of the differential $d : \mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathbb{A}^n|Y}$. Furthermore $Hom(\Omega_{\mathbb{A}^n|Y}, \mathcal{O}_Y) \cong A^n$ and $Hom(\mathcal{I}/\mathcal{I}^2) \cong A$. We can write the map $Hom(\Omega_{\mathbb{A}^n|Y}, \mathcal{O}_Y) \to Hom(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$ as

$$\phi : A^n \to A,\ (\alpha_1, ..., \alpha_n) \mapsto \alpha_1 \frac{\partial f}{\partial x_1} + ... + \alpha_n \frac{\partial f}{\partial x_n}.$$

We rewrite our exact sequence as

$$0 \mapsto Hom(\Omega_Y, \mathcal{O}_Y) \to A^n \to A \to Ext^1(\Omega_Y, \mathcal{O}_Y) \mapsto 0.$$

Then $Im(\phi) = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}) \subseteq A$, and $Ext^1(\Omega_Y, \mathcal{O}_Y) \cong A/(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})$.
Now let $Y = C \subseteq \mathbb{A}^2$ be a nodal curve. In a *é*tale neighborhood of the node we can assume $C = Spec(A)$, where $A = k[x, y]/(xy)$. From the preceding discussion we get $Ext^1(\Omega_C, \mathcal{O}_C) \cong A/(x, y) \cong k$. So $Ext^1(\Omega_C, \mathcal{O}_C)_p = 0$ if p is a smooth point of C and $Ext^1(\Omega_C, \mathcal{O}_C)_p = k$ if $p \in Sing(C)$. Furthermore

$$\mathcal{E}xt^1(\Omega_C, \mathcal{O}_X) \cong \sum_{p \in Sing(C)} \mathcal{O}_p.$$

6. Curves of Genus One

An *elliptic curve* over an algebraically closed field is a smooth projective curve of genus one. Let X be an elliptic curve and let $P \in X$ be a point, consider the linear system $|2P|$ on X. Since the curve is not rational $|2P|$ has no base points, and since $deg(K-2P) = 2g-2-2 = -2 < 0$ the divisor $|2P|$ is nonspecial i.e. $h^0(K - 2P) = 0$. By Riemann-Roch theorem $h^0(2P) = deg(2P) - g + 1 = 2$. Then the linear system $|2P|$ defines a morphism $f : X \to \mathbb{P}^1$ of degree 2 on $\mathbb{P}^1$. Now by Riemann-Hurwitz theorem we have

$$2g - 2 = deg(f)(2g_{\mathbb{P}^1} - 2) + deg(R_f),$$

then $deg(R_f) = 2 \cdot deg(f) = 4$, and f is ramified in four points and clearly P is one of them. If x_1, x_2, x_3, ∞ are the four branch points in $\mathbb{P}^1$, then there is a unique automorphism of $\mathbb{P}^1$ sending x_1 to 0, x_1 to 1, and leaving ∞ fixed, namely $y = \frac{x-x_1}{x_2-x_1}$. After this change of coordinates we can assume that f is branched over $0, 1, \lambda, \infty \in \mathbb{P}^1$, whit $\lambda \in k,\ \lambda \neq 0, 1$.
We define the j-invariant of the elliptic curve X by

$$j = j(\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$

It is well known that over an algebraically closed field k with $char(k) \neq 2$ the scalar $j(X)$ depends only on X. Furthermore two elliptic curves $X, X^{'}$ are isomorphic if and only if $j(X) = j(X^{'})$, and every element of k is the j-invariant of some elliptic curve. Then there is a one-to-one correspondence with the set of elliptic curves up to isomorphism and $\mathbb{A}^1_k$ given by $X \mapsto j(X)$.

Definition 6.1. A family of elliptic curves over a scheme S is a flat morphism of schemes $X \to S$ whose fibers are smooth curves of genus one, with a section $\sigma : S \to X$. In particular, an elliptic curve is a smooth curve C of genus one with a rational point $P \in C$.

Consider the functor $\mathcal{F} : \mathfrak{Sch} \to \mathfrak{Sets}$ where $\mathcal{F}(S)$ is the set of families of elliptic curves over S modulo isomorphism. One can prove that $\mathcal{F}$ does not have a fine moduli space, but the affine line $\mathbb{A}^1_k$ is a coarse moduli space for $\mathcal{F}$.
Now a natural question is how to compactify this coarse moduli space to obtain a complete moduli space. In addition to elliptic curves we admit also irreducible nodal curve of arithmetic genus $p_a = 1$ with a fixed nonsingular point. We consider families $X \to S$ whose fibers are elliptic curves or pointed nodal curve, then taking $j(C) = \infty$ for the nodal curve the projective line $\mathbb{P}^1$ becomes a coarse moduli moduli space.
Let C be a reduced irreducible curve with $p_a = 1$ and such that $Sing(C)$ is a node. A such curve can be embedded in $\mathbb{P}^2$ has the nodal cubic $C = Z(y^2z - x^3 + x^2z)$. Consider the low degrees exact sequence for Ext functors,

$$0 \mapsto H^1(X, \mathcal{H}om(\Omega_C, \mathcal{O}_C)) \to Ext^1(\Omega_C, \mathcal{O}_C) \to H^0(X, \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)) \to H^2(X, \mathcal{H}om(\Omega_C, \mathcal{O}_C)).$$

Since $\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)$ is concentrated at the singular point of C we know that $H^0(X, \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C))$ is a 1-dimensional k-vector space. Now we consider the sheaf $\mathcal{H}om(\Omega_C, \mathcal{C}) = T_C$.
Recall that if X is a smooth variety and $Y \subseteq X$ is a closed irreducible subscheme defined by the sheaf of ideals $\mathcal{I}$, then there is an exact sequence

$$\mathcal{I}/\mathcal{I}^2 \to \Omega_X \otimes \mathcal{O}_Y \to \Omega_Y \mapsto 0.$$

Furthermore Y is smooth if and only if

- the sheaf Ω_Y is locally free, and
- the sequence above is exact on the left also:

$$0 \mapsto \mathcal{I}/\mathcal{I}^2 \to \Omega_X \otimes \mathcal{O}_Y \to \Omega_Y \mapsto 0.$$

Consider the sequence for a general subscheme Y and apply the functor $\mathcal{H}om(-, \mathcal{O}_Y)$. We obtain

$$0 \mapsto T_Y \to T_{X|Y} \to N_{Y/X} \to \mathcal{E}xt^1(\Omega_Y, \mathcal{O}_Y) \mapsto 0.$$

For our nodal curve C in $\mathbb{P}^2$ we have

$$0 \mapsto T_C \to T_{\mathbb{P}^2|C} \to N_{C/\mathbb{P}^2} \to \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C) \mapsto 0.$$

We know that $N_{C/\mathbb{P}^2} = \mathcal{O}_C(C) = \mathcal{O}_C(3)$, let D be the divisor associated to $\mathcal{O}_C(3)$. Since C is a local complete intersection the dualizing sheaf ω° is an invertible sheaf. We define the canonical divisor as the divisor corresponding to ω° with support in C_{reg}. Since there are no regular differentials on C we have $deg(K-D) < 0$. By Riemann-Roch theorem for singular curves we get

$$h^0(N_{C/\mathbb{P}^2}) = deg(D) + 1 - p_a = 9 + 1 - 1 = 9.$$

Consider now the Euler sequence

$$0 \mapsto \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3} \to T_{\mathbb{P}^2} \mapsto 0.$$

Tensorizing by $\mathcal{O}_C$ we get

$$0 \mapsto \mathcal{O}_C \to \mathcal{O}_C(1)^{\oplus 3} \to T_{\mathbb{P}^2|C} \mapsto 0.$$

Using the dualizing sheaf $\omega_C^\circ \cong \mathcal{O}_C$, and Serre duality we get $h^1(\mathcal{O}_C(1)) = h^0(\mathcal{O}_C(-1)) = 0$. The cohomology sequence looks like

$$0 \mapsto H^0(C, \mathcal{O}_C) \to H^0(C, \mathcal{O}_C(1)^{\oplus 3}) \to H^0(C, T_{\mathbb{P}^2|C}) \to H^1(C, \mathcal{O}_C) \mapsto 0,$$

so $h^0(T_{\mathbb{P}^2|C}) = 9$. Furthermore the map $H^0(C, N_{C/\mathbb{P}^2}) \to H^0(C, \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C))$ is surjective since the former parametrizes the embedded deformations of C as a subscheme of $\mathbb{P}^2$ and the latter parametrizes the abstract deformations of the node. We conclude that $h^0(T_C) > 0$. Let $\sigma \in H^0(C, T_C)$ be a nonzero section, we have an exact sequence $0 \mapsto \mathcal{O}_C \xrightarrow{\sigma} T_C \to R \mapsto 0$. The cokernel R is not zero, because T_C is not locally free. Then $T_C^\vee$ is a proper subsheaf of $\mathcal{O}_C$, using the dualizing sheaf $\omega_C^\circ \cong \mathcal{O}_C$ and Serre duality we get $h^1(T_C) = h^0(T_C^\vee) = 0$. We conclude that $Def(C)$ is one-dimensional.

7. The Moduli Stack $\overline{\mathfrak{M}_g}$

The search for a Moduli Space M_g for the curves of genus $g \geq 2$ is a classical problem in algebraic geometry. Riemann computed its dimension $dim(\mathfrak{M}_g) = 3g - 3$.
By Riemann-Hurwitz formula to any collection of $2d + 2g - 2$ points on $\mathbb{P}^1$ corresponds a curve X with a finite morphism $\phi : X \to \mathbb{P}^1$ of degree d. To give such a morphism is equivalent to choose a divisor D of degree d on X (i.e. d distinct points on X) and a element in $H^0(X, \mathcal{O}_X(D))$. If we consider divisors of degree $d > 2g - 2$, by Riemann-Roch we get $h^0(D) = d - g + 1$. Then we have to subtract $dim(Aut(X))$ but a curve of genus $g \geq 2$ as only a finite number of automorphism. We conclude that

$$dim(\mathfrak{M}_g) = 2d + 2g - 2 - (d + d - g + 1) = 3g - 3.$$

Deligne and Mumford introduced a compactification of the moduli space allowing singular stable curves, and in this context they hinted a new object, the moduli stack.
We consider projective smooth curves of genus $g \geq 2$ over an algebraically closed field k, a such curve have only a finite number of automorphisms. Consider the functor $\mathcal{F} : \mathfrak{Sch} \to \mathfrak{Sets}$ such that $\mathcal{F}(S)$ is the sets of flat families $X \to S$ whose fibers are curves of genus g, up to isomorphism. We say that two families $X \xrightarrow{\pi} S$, $X^{'} \xrightarrow{\pi^{'}} S^{'}$ are isomorphic if there exists an isomorphism of schemes $f : X \to X^{'}$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & X^{'} \arrow[dl, "\pi^{'}"] \\ & S & \end{tikzcd}$$

One of the main theorem of the theory is the following.

Theorem 7.1. *The functor $\mathcal{F}$ of curves of genus $g \geq 2$ over an algebraically closed field k has a coarse moduli space $\mathfrak{M}_g$, which is a normal quasi-projective variety of dimension $3g-3$.*

For a proof see Mumford's book *Geometric Invariant Theory*, Deligne and Mumford article *The irreducibility of the space of curves of given genus*, and Fulton article *On the irreducibility of the moduli space of curves.*

7.1. **Automorphisms of Curves.** The only curve of genus one is $\mathbb{P}^1$, ant its automorphisms group is $PGL(2)$ which is an open subset of $\mathbb{P}^3$. If we choose one or two marked points in $\mathbb{P}^1$ the automorphisms group remains infinite of dimension two and one respectively. However a well known theorem in projective geometry asserts that if we fix three marked points the automorphisms group has only one element.
We will see that an elliptic curve has infinitely many automorphisms, but if we choose a marked point on the elliptic curve the its automorphisms group is finite. Finally we will prove that any curve X of genus $g \geq 2$ has finitely many automorphisms, and we will give a bound on the cardinality on $Aut(X)$.
Recall that an elliptic curve X has a group structure, more precisely if we fix a point on X then we get a bijective correspondence between the points of X and the divisors of degree zero in $Cl^0(X)$, so any translation $X \times X \to X$ gives an automorphism of X. Clearly if we choose a marked point $p \in X$, then the only possible translation is the identity, in this way the automorphism group becomes finite. For more details see *J.H. Silverman - The Arithmetic of Elliptic Curves.*

Proposition 7.2. *Let E be an elliptic curve over k with a marked point. The automorphisms group $Aut(E)$ is a finite group of order dividing 24. More precisely*

- *if $j(E) \neq 0, 1728$, then $|Aut(E)| = 2$,*
- *if $j(E) = 1728$ and $chat(k) \neq 2, 3$, then $|Aut(E)| = 4$,*
- *if $j(E) = 0$ and $chat(k) \neq 2, 3$, then $|Aut(E)| = 6$,*
- *if $j(E) = 0, 1728$ and $chat(k) = 3$, then $|Aut(E)| = 12$,*
- *if $j(E) = 0, 1728$ and $chat(k) = 2$, then $|Aut(E)| = 24$.*

Proof. We consider the case $char(k) \neq 2, 3$. Then E can be realized as a plane smooth cubic and can be written in Weierstrass form

$$y^2 = x^3 + \alpha x + \beta,$$

furthermore every automorphism of E is of the form

$$x = u^2 x^{'},\ y = u^3 y^{'},$$

for some $u \in \underline{k}^*$. Such a substitution will give an automorphism if and only if

$$u^{-4}\alpha = \alpha,\ u^{-6}\beta = \beta.$$

If $\alpha \cdot \beta = 0$ then $j(E) \neq 0, 1728$, the only possibilities are $u = \pm 1$. If $\beta = 0$ then $j(E) = 1728$, and u satisfies $u^4 = 1$, so $Aut(E)$ is cyclic of order 4. If $\alpha = 0$ then $j(E) = 0$, and u satisfies $u^6 = 1$, so $Aut(E)$ is cyclic of order 6. □

Proposition 7.3. *Any smooth curve X of genus $g \geq 2$ has finitely many automorphisms.*

Before proving the proposition we recall some general facts about canonically embedded varieties.

Remark 7.4. (*Canonically Embedded Varieties*) Let $f : X \to Y$ be a dominant morphism between smooth varieties. The pullback $f^* : f^*\Omega_Y \to \Omega_X$ defines a canonical morphisms between the cotangent sheaves, and since pullback commutes with maximal exterior powers we get a canonical morphism $f^* : f^*\omega_Y \to \omega_X$ of the canonical sheaves. In particular if $X = Y$ and $f \in Aut(X)$, since $f^*\omega_X \cong \omega_X$, we get an automorphism f^* of ω_X. Then an automorphism of X induces an automorphism of ω_X, and an automorphism on the vector space of the its global section $H^0(X, \omega_X)$. Suppose now that ω_X is ample, then $\omega_X^{\otimes n}$ is very ample for some $n \geq 0$. Any automorphism of X induces also an automorphism of $\omega_X^{\otimes n}$. Let $\phi : X \to \mathbb{P}(H^0(X, \omega_X^{\otimes n})^*)$ be the corresponding embedding. Then we have an action of $Aut(X)$ on $\mathbb{P}(H^0(X, \omega_X^{\otimes n})^*)$, and any $f \in Aut(X)$ induces an automorphism of $\mathbb{P}(H^0(X, \omega_X^{\otimes n})^*) = \mathbb{P}^N$. We have seen that if X has ample canonical sheaf then $Aut(X)$ is a closed algebraic subgroup of $PGL(N+1)$. Clearly the same argument works if X has anticanonical ample sheaf.

Proof. Recall that if $f : X \to Y$ is a morphism of schemes, with X separated and Y smooth, and Def_f is the deformation functor of f, then $T^1 Def_f = H^0(X, f^*T_Y)$. In particular for $f = Id_X : X \to X$ we get $T^1_{Id_X} Def_{Id_X} = T_{Id_X} Aut(X) = H^0(X, T_X)$, and $h^0(X, T_X) = 0$ since X is a curve of genus $g \geq 2$. The curve X has canonical ample sheaf, and by the preceding remark we can embed $Aut(X)$ in $PGL(N+1) \subseteq \mathbb{P}^{(N+1)^2-1}$ as closed subscheme. Since the tangent space of $Aut(X)$ has dimension zero we conclude that $Aut(X)$ is a finite set of points. □

In the following proposition we give a bound on the number of automorphisms of a curve of genus $g \geq 2$.

Proposition 7.5. *Let X be a projective curve of genus $g \geq 2$, then the group $Aut(X)$ is finite and $|Aut(X)| \leq 84(g-1)$.*

Proof. Let $W(X)$ be the set of Weierstrass points of X, we know that $W(X)$ is finite. If $\phi \in Aut(X)$ is a non trivial automorphism then ϕ as at most $2g+2$ fixed points. Since the set of Weierstrass points is fixed by the group $Aut(X)$ we have a morphism

$$F : Aut(X) \to Perm(W(X)),$$

where $Perm(W(X))$ is the group of permutations of $W(X)$. If X is non hyperelliptic there are more that $2g+2$ Weierstrass points on X and there a unique automorphism that leaves more that $2g+2$ points fixed, the identity. So $Ker(F) = \{Id_X\}$.

If X is hyperelliptic then any automorphism in the subgroup (J) generated by the involution $J : X \to X$ fixes the Weierstrass points, but since $J^2 = Id_X$ this subgroup is finite. We conclude that F is a morphism of $Aut(X)$ into a finite group and with finite kernel, then the group $Aut(X)$ is finite.

Let $G = Aut(X)$ and $|G| = n$, consider the projection $\pi : X \to X/G$. For any $\overline{x} \in X/G$ we have $\pi^{-1}(\overline{x}) = \{x \in X \mid \pi(x) = \overline{x}\} = \{x \in X \mid \exists\, g \in G,\ g(x) = \overline{x}\} = \{g^{-1}(\overline{x}),\ g \in G\}$, then π is a morphism of degree n. The map π is branched only at fixed point of G. Let $P_1, ..., P_s$ be a maximal sets of ramification points of X lying over distinct points of X/G, and let r_i be the index of ramification of P_i. Recall that if $P \in X$ is a ramification point, and r is its ramification index, then the fiber $\pi^{-1}(\pi(P))$ consists of exactly $\frac{n}{r}$ points, each having ramification index r, essentially because X is a covering space for X/G. So in the fiber of any P_j there are $\frac{n}{r_j}$ points each with ramification index r_j. Then the degree of the ramification divisor is

$$deg(R_\pi) = \sum_{j=1}^{s}(r_j - 1)\frac{n}{r_j} = n\sum_{j=1}^{s}(1 - \frac{1}{r_j}).$$

By Riemann-Hurwitz formula we get $2g-2 = n(2\alpha - 2) + n\sum_{j=1}^{s}(1-\frac{1}{r_j})$, where α is the genus of X/G. Then

$$\frac{2g-2}{n} = 2\alpha - 2 + \sum_{j=1}^{s}(1 - \frac{1}{r_j}).$$

Note that since $r_j \geq 2$ we have $\frac{1}{2} \leq 1 - \frac{1}{r_j} < 1$. Since we may assume $n > 1$ it is clear that $g > \alpha$. Now we have to analyze the expression $2\alpha - 2 + \sum_{j=1}^{s}(1-\frac{1}{r_j})$.

(1) If $\alpha \geq 2$ we obtain $2\alpha - 2 + \sum_{j=1}^{s}(1-\frac{1}{r_j}) \geq 2 - \sum_{j=1}^{s}(1-\frac{1}{r_j}) \geq 2$, so $\frac{2g-2}{n} \geq 2$ and
$$n \leq g-1.$$

(2) if $\alpha = 1$ then $2\alpha - 2 + \sum_{j=1}^{s}(1-\frac{1}{r_j}) = \sum_{j=1}^{s}(1-\frac{1}{r_j}) \geq \frac{1}{2}$, so $\frac{2g-2}{n} \geq \frac{1}{2}$ and
$$n \leq 4(g-1).$$

(3) if $\alpha = 0$ then $2\alpha - 2 + \sum_{j=1}^{s}(1-\frac{1}{r_j}) = \sum_{j=1}^{s}(1-\frac{1}{r_j}) - 2$. Since $\sum_{j=1}^{s}(1-\frac{1}{r_j}) - 2 > 0$ and $1 - \frac{1}{r_j} < 1$, we conclude that $s \geq 3$.
   - If $s \geq 5$, then $\sum_{j=1}^{s}(1-\frac{1}{r_j}) - 2 \geq \frac{1}{2}$, so $\frac{2g-2}{n} \geq \frac{1}{2}$ and
   $$n \leq 4(g-1).$$
   - If $r = 4$ then the r_j cannot be all equal to 2, otherwise we will have $\frac{2g-2}{n} = 0$, so $g = 1$. Then at least one is ≥ 3 and gives$\sum_{j=1}^{s}(1-\frac{1}{r_j}) - 2 \geq 3(1-\frac{1}{2}) + (1-\frac{1}{3}) - 2 = \frac{1}{6}$, so $\frac{2g-2}{n} \geq \frac{1}{6}$ and
   $$n \leq 12(g-1).$$
   - In the case $s = 3$ we can assume without loss of generality $2 \leq r_1 \leq r_2 \leq r_3$. We have $r_3 > 3$ otherwise $\sum_{j=1}^{s}(1-\frac{1}{r_j}) - 2 < 0$. Then $r_2 \geq 3$.
   If $r_3 \geq 7$ then $n \leq 84(g-1)$.
   If $r_3 = 6$ and $r_1 = 2$ then $r_2 \geq 4$ and $n \leq 24(g-1)$.
   If $r_3 = 6$ and $r_1 \geq 3$ then $n \leq 12(g-1)$.
   If $r_3 = 5$ and $r_1 = 2$ then $r_2 \geq 4$ and $n \leq 40(g-1)$.
   If $r_3 = 5$ and $r_1 \geq 3$ then $n \leq 15(g-1)$.
   If $r_3 = 4$ then $r_1 \geq 3$ and $n \leq 24(g-1)$.

□

To compactify the coarse moduli space $\mathfrak{M}_g$ Deligne and Mumford introduces *stable* curves. We have seen that $T_{Id_X}Aut(X) = H^0(X, T_X)$, an element of this space is called an *infinitesimal automorphism*.

Definition 7.6. A reduced, connected, projective curve X, having at most nodes as singularities is said to be stable if $H^0(X, T_X) = 0$, i.e. X has no infinitesimal automorphisms.

Clearly for a curve X of genus $g \geq 2$ the following are equivalent,

- X has no infinitesimal automorphisms,
- $H^0(X, T_X) = 0$,

- $Aut(X)$ is finite.

By the preceding discussion any smooth curve of genus $g \geq 2$ is stable.

Consider the local infinitesimal deformation functor of $\mathcal{F}$ for a stable curve X of genus $g \geq 2$,

$$Def_X : \mathfrak{Art} \to \mathfrak{Sets},$$

which associates to any Artin local algebra A the set of isomorphism classes $\Upsilon \to Spec(A)$ of families of curves of genus g over $Spec(A)$, with a fixed isomorphism $\Upsilon_0 \to X$, where $\Upsilon_0 \to Spec(k)$ is the central fiber of Υ. Note that the isomorphism $\Upsilon_0 \to X$ is not unique, indeed we can recover any other isomorphism composing with an automorphism of X, and the set of such isomorphisms is a principal homogeneous space under the action of $Aut(X)$ .We denote by $\overline{\mathfrak{M}_g}$ the Deligne-Mumford compactification of $\mathfrak{M}_g$. Then we can think the points of $\mathfrak{M}_g$ as isomorphism classes of curves of genus g, and the points of the boundary $\overline{\mathfrak{M}_g} \setminus \mathfrak{M}_g$ as isomorphism classes of nodal stable curves of arithmetic genus g. Our aim is to prove that $\overline{\mathfrak{M}_g}$ using deformation theory. The following remark will be important in the proof of Smoothness of $\overline{\mathfrak{M}_g}$.

Remark 7.7. Let X be a proper scheme and let Def_X be its deformation functor. Then $T^i_{Def_X} = Ext^i(L^\bullet_X, \mathcal{O}_X)$, where $L^\bullet_X$ is the cotangent complex of X. If X has only local complete intersection singularities the $L^\bullet_X$ coincides with Ω_X in degree zero. Recall that from the spectral sequence of Ext groups we have

$$H^q(X, \mathcal{E}xt^p(\Omega_X, \mathcal{O}_X)) \Rightarrow Ext^{p+q}(\Omega_X, \mathcal{O}_X).$$

Consider the special case where $X = C$ is a nodal curve and $p + q = 2$. Then

- $H^0(C, \mathcal{E}xt^2(\Omega_C, \mathcal{O}_C)) = 0$ because Ω_C admits a locally free resolution of length one. Indeed take an embedding $C \to Y$ of Y in a smooth surface, then we have an exact sequence
$$0 \mapsto \mathcal{I}/\mathcal{I}^2 \to \Omega_Y \otimes \mathcal{O}_C \to \Omega_C \mapsto 0.$$
- $H^1(C, \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)) = 0$ because $\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)$ is supported on $Sing(C)$ which is zero dimensional.
- $H^2(C, \mathcal{H}om(\Omega_C, \mathcal{O}_C)) = 0$ because $dim(C) = 1$.

We conclude that $Ext^2(\Omega_C, \mathcal{O}_C) = T^2 Def_C = 0$.

Theorem 7.8. *(Smoothness of $\overline{\mathfrak{M}_g}$) Let X be a stable curve of arithmetic genus $g \geq 2$. Then the functor of local infinitesimal deformations Def_X of X is pro-representable by a regular local ring of dimension $3g-3$. In other words $\overline{\mathfrak{M}_g}$ is a smooth Deligne-Mumford stack of dimension*

$$dim(\overline{\mathfrak{M}_g}) = 3g - 3.$$

Proof. The functor Def_X is pro-representable since X is projective and does not have infinitesimal automorphism. Furthermore $T^2 Def_X = H^2(X, T_X) = 0$ since $dim(X) = 1$, then there are no obstructions to deforming X and the local ring representing Def_X is regular. Furthermore from remark 7.7 we get $Ext^2(\Omega_X, \mathcal{O}_X) = T^2 Def_X = 0$ for a nodal curve. Then in any case the deformation functor of X is unobstructed. So far we have proved that $\overline{\mathfrak{M}_g}$ is representable by a smooth Deligne-Mumford stack. To compute its dimension we distinguish two cases.

(1) If X is a smooth curve, and $0 \mapsto I \to A \to B \mapsto 0$ is a semi-small exact sequence in $\mathfrak{Art}$, then there is a functorial exact sequence
$$H^1(X, T_X) \otimes I \to Def_X(A) \to Def_X(B) \to H^2(X, T_X) \otimes I.$$
On a curve $T_X = \omega_X\check{}$, where ω_X is the canonical sheaf of X. Then $deg(T_X) = 2 - 2g$, and since $h^0(T_X) == 0$, by Riemann-Roch theorem we get $h^0(T_X) - h^1(T_X) = 2 - 2g - g + 1 = 3 - 3g$, and $h^1(T_X) = 3g - 3$. We conclude that in a point $x \in \overline{\mathfrak{M}_g}$ corresponding to the isomorphism class of a smooth curve X, the tangent space $T_x\overline{\mathfrak{M}_g}$ has dimension $3g - 3$.

(2) Now consider the case where X is a stable nodal curve. We have a sequence

$$0 \mapsto H^1(X, \mathcal{H}om(\Omega_X, \mathcal{O}_X)) \to Ext^1(\Omega_X, \mathcal{O}_X) \to H^0(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) \mapsto 0,$$

there being no H^2 on a curve. We denote by δ the number of nodes in X. Since the sheaf Ω_X is locally free on the smooth locus of X, the sheaf $\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X))$ is just k at each node, then $dim(H^0(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X))) = \delta$. The curve X is l.c.i, then the dualizing sheaf ω_X is an invertible sheaf, and since $\omega_X \cong \Omega_X$ on the open set of regular points, we have an injective morphism $\check{\omega_X} \to \mathcal{H}om(\Omega_X, \mathcal{O}_X)$, and an exact sequence

$$0 \mapsto \check{\omega_X} \to \mathcal{H}om(\Omega_X, \mathcal{O}_X) \to \mathcal{O}_Z \mapsto 0,$$

where $Z = Sing(X)$. Since X is stable $h^0(\mathcal{H}om(\Omega_X, \mathcal{O}_X)) = 0$, by the cohomology exact sequence we get $h^0(\check{\omega_X}) = 0$, and

$$0 \mapsto H^0(X, \mathcal{O}_Z) \to H^1(X, \check{\omega_X}) \to H^1(\mathcal{H}om(\Omega_X, \mathcal{O}_X)) \mapsto 0.$$

By Riemann-Roch for singular curves we get $h^1(\check{\omega_X}) = 3g-3$, and since $h^0(\mathcal{O}_Z) = \delta$ we get $h^1(\mathcal{H}om(\Omega_X, \mathcal{O}_X)) = 3g-3-\delta$. Finally

$$dim(Ext^1(\Omega_X, \mathcal{O}_X)) = h^1(T_X) + h^0(\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) = 3g-3-\delta+\delta = 3g-3.$$

We conclude that any point of $\overline{\mathfrak{M}_g}$ is smooth and $\overline{\mathfrak{M}_g}$ is a smooth stack of dimension $3g-3$. □

7.2. **Stacks.** The study of moduli problems for elliptic curves and curves of genus $g \geq 2$ introduces a new kind of objects: the so called moduli stacks. We have seen that a moduli problem gives rise to a functor, if the functor is representable we have a fine moduli space, that is a scheme. Sometimes, if it is not representable one can find a coarse moduli space, which tells us the isomorphism classes of our objects over a field, but does not describe all the possible families of objects. It happens that the functor related to a moduli problem is not representable by a scheme. We search for a sort of generalized scheme.

A scheme is constructed out of affine schemes by gluing the isomorphism defined on Zariski open subset. In the same spirit consider a collection of schemes $\{X_i\}$, and for each i, j *é*tale morphisms $Y_{i,j} \to X_i$, $Y_{j,i} \to X_j$ and isomorphisms $\phi_{i,j} : Y_{i,j} \to Y_{j,i}$, satisfying a cocycle condition for each i, j, k. We glue together the X_i along the $\phi_{i,j}$. This quotient may not exist in the category of schemes, but it is an *algebraic space*.

Instead of the functor $\mathcal{F}$, which sends any scheme S in the set of isomorphism classes of families $X \to S$, consider a new object $\mathcal{F}$, which to each scheme S assigns the category $\mathcal{F}(S)$ of families and isomorphisms between such families. This object is called a fibered category over the category of schemes. The sheaf axioms for the functor $\mathcal{F}$ are replaced by the *stack axioms* for the fibered category $\mathcal{F}$, which are the following. For any scheme S and any *é*tale covering $\{U_i \to S\}$, consider

$$\mathcal{F}(S) \to \prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \times_S U_j) \rightrightarrows \prod \mathcal{F}(U_i \times_S U_j \times_S U_k).$$

- The fact that the first arrow is injective means that if $a, b \in \mathcal{F}(S)$ and if a_i, b_i are their restriction on $\mathcal{F}(U_i)$, and there is an isomorphism $\phi_i : a_i \to b_i$ such that for each i, j the isomorphisms ϕ_i, ϕ_j restrict to the same isomorphism of $a_{i,j}$ and $b_{i,j}$ on $U_i \times_S U_j$, then there is a unique isomorphism ϕ inducing ϕ_i on each U_i.
- The fact that the sequence is exact at the first middle term means that if we give objects $a_i \in \mathcal{F}(U_i)$ for each i and isomorphisms $\phi_{i,j} : a_i \to a_j$ on $U_i \times_S U_j$ satisfying a cocycle condition on each $U_i \times_S U_j \times_S U_k$, then there exists a unique object $a \in \mathcal{F}(S)$ restricting to each a_i on U_i.

A *Deligne-Mumford stack* is a fibered category $\mathcal{F}$ satisfying the stack axioms, and such that there exists a scheme X and a surjective *é*tale morphism $Hom(-, X) \to \mathcal{F}$. An *Artin stack* is the same but require that only that $Hom(-, X) \to \mathcal{F}$ be smooth.

The moduli space of curves $\overline{\mathfrak{M}_g}$ is a Deligne-Mumford stack for any $g \geq 2$. In the paper *The irreducibility of the space of curves of given genus*, Deligne and Mumford introduced stacks for the first time, they compactified the stack $\mathfrak{M}_g$ adding stable curves, and they prove its irreducibility in any characteristic.

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